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INVARIANT MEASURES FOR CERTAIN EXPANSIVE $\mathbb{Z}^2$-ACTIONS

BY

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ABSTRACT

Let $p > 1$ be prime, and let $Y \subset X = (\mathbb{Z}/p\mathbb{Z})^{\mathbb{Z}^2}$ be an infinite, closed, shift-invariant subgroup with the following properties: the restriction to Y of the shift-action σ of $\mathbb{Z}^2$ on X is mixing with respect to the Haar measure λ_Y of Y, and every closed, shift-invariant subgroup $Z \subsetneq Y$ is finite. We prove that every sufficiently mixing, non-atomic, shift-invariant probability measure μ on Y is equal to λ_Y.

Let $G \neq \{0\}$ be a finite, abelian group, and let σ be the shift-action $(\sigma_{\mathbf{m}}(x))_{\mathbf{n}} = x_{\mathbf{m}+\mathbf{n}}$ of $\mathbb{Z}^2$ on $X = G^{\mathbb{Z}^2}$. Then there exist uncountably many distinct, σ-invariant probability measures on X which are Bernoulli—and hence mixing of every order—under σ: indeed, let ν be a probability measure on G, put $\mu = \nu^{\mathbb{Z}^2}$, and note that different measures ν, ν' lead to inequivalent Bernoulli measures $\nu^{\mathbb{Z}^2}$ and $\nu'^{\mathbb{Z}^2}$. However, if $Y \subset X$ is a closed, shift-invariant subgroup such that the restriction σ^Y of σ to Y is mixing (with respect to the normalized Haar measure λ_Y of Y), then λ_Y may be the only non-atomic, σ^Y-invariant, mixing probability measure, although there always exist many different invariant, non-atomic, ergodic, *non-mixing* probability measures on Y. In order to study this phenomenon we assume that $G = \mathbb{Z}/p\mathbb{Z}$ is a cyclic group of prime order p, and consider subgroups $Y \subset X^{(p)} = (\mathbb{Z}/p\mathbb{Z})^{\mathbb{Z}^2}$ with the following properties.

(a) Y is infinite, closed, and shift-invariant;

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(b) Every closed, shift-invariant subgroup $Z \subsetneq Y$ is finite;

(c) σ^Y is mixing with respect to λ_Y.

1. *Definition:* Let $Y \subset X^{(p)} = (\mathbb{Z}/p\mathbb{Z})^{\mathbb{Z}^2}$ be a subgroup satisfying (a)–(c), and let μ be a shift-invariant probability measure on Y. A finite, non-empty set $E \subset \mathbb{Z}^2$ is **μ-mixing** if

$$\lim_{k\to\infty} \mu\left(\bigcap_{\mathbf{n}\in E} \sigma^Y_{-k\mathbf{n}}(B(\mathbf{n}))\right) = \prod_{\mathbf{n}\in E} \mu(B(\mathbf{n}))$$

for every map $\mathbf{n} \mapsto B(\mathbf{n})$ from E into the σ-algebra $\mathfrak{B}_Y$ of Borel subsets of Y, and **μ-non-mixing** otherwise. A μ-non-mixing set $E \subset \mathbb{Z}^2$ is **minimal μ-non-mixing** if every set E' with $\varnothing \neq E' \subsetneq E$ is μ-mixing.

Every λ_Y-non-mixing set is also μ-non-mixing (cf. Lemma 4). If the converse is true, then the following theorem shows that μ has to be equal to λ_Y; in other words, λ_Y is the unique 'most mixing' measure for σ^Y.

2. THEOREM: *Let $p > 1$ be a rational prime, $Y \subset X^{(p)}$ a subgroup satisfying* (a)–(c), *and let μ be a non-atomic, shift-invariant probability measure on Y. If there exists a minimal λ_Y-non-mixing set $E \subset \mathbb{Z}^2$ which is also minimal μ-non-mixing, then $\mu = \lambda_Y$.*

For the proof of Theorem 2 we need an explicit description of the subgroups $Y \subset X^{(p)}$ satisfying (a)–(c). Let $\mathfrak{R}_2^{(p)} = \mathbb{F}_p[u_1^{\pm 1}, u_2^{\pm 1}]$ be the ring of Laurent polynomials in the variables u_1, u_2 with coefficients in the prime field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, and write a typical element $f \in \mathfrak{R}_2^{(p)}$ as $f = \sum_{\mathbf{n}\in\mathbb{Z}^2} c_f(\mathbf{n}) u^{\mathbf{n}}$, where $c_f(\mathbf{n}) \in \mathbb{F}_p$ and $u^{\mathbf{n}} = u_1^{n_1} u_2^{n_2}$ for every $\mathbf{n} = (n_1, n_2) \in \mathbb{Z}^2$. An element $f \in \mathfrak{R}_2^{(p)}$ is a generalized polynomial in a single variable if its support $\mathcal{S}(f) = \{\mathbf{n} \in \mathbb{Z}^2 \colon c_f(\mathbf{n}) \neq 0\}$ is contained in the line $\{\mathbf{k} + m\mathbf{l} \colon m \in \mathbb{Z}\}$ for some $\mathbf{k}, \mathbf{l} \in \mathbb{Z}^2$, $\mathbf{l} \neq \mathbf{0}$. The dual group $\widehat{X^{(p)}}$ of $X^{(p)}$ can be identified with $\mathfrak{R}_2^{(p)}$ by setting $\langle f, x\rangle = e^{\frac{2\pi i}{p}\sum_{\mathbf{n}\in\mathbb{Z}^2} c_f(\mathbf{n}) x_{\mathbf{n}}}$ for every $f \in \mathfrak{R}_2^{(p)}$ and $x = (x_{\mathbf{n}}) = (x_{\mathbf{n}}, \mathbf{n} \in \mathbb{Z}^2) \in X^{(p)}$, where the element $\sum_{\mathbf{n}\in\mathbb{Z}^2} c_f(\mathbf{n}) x_{\mathbf{n}} \in \mathbb{F}_p$ is identified with the corresponding integer in $\{0, \dots, p-1\}$. If $Y \subset X^{(p)}$ is a closed, shift-invariant subgroup, then $\mathfrak{a} = Y^{\perp} \subset \mathfrak{R}_2^{(p)} = \widehat{X^{(p)}}$ is an ideal; conversely, if $\mathfrak{a} \subset \mathfrak{R}_2^{(p)}$ is an ideal, then $Y = \widehat{\mathfrak{R}_2^{(p)}/\mathfrak{a}}$ is a closed, shift-invariant subgroup of $X^{(p)}$. The following lemma is an immediate consequence of the Propositions 2.12–2.13 in [Kit-Sch1] and Theorem 3.5 in [Sch].

3. LEMMA: *Let $\{0\} \neq Y \subset X^{(p)}$ be a closed, shift-invariant subgroup, and let $\mathfrak{a} = Y^{\perp} \subset \widehat{X^{(p)}} = \mathfrak{R}_2^{(p)}$ be the annihilator of Y. Then Y satisfies* (a)–(c) *if and*

only if the ideal $\mathfrak{a} \subset \mathfrak{R}_2^{(p)}$ is non-zero, prime, and principal, and not generated by a generalized polynomial in a single variable.

Lemma 3 yields an abundance of subgroups $Y \subset X^{(p)}$ satisfying (a)–(c). Let $Y \subset X^{(p)}$ be such a subgroup, and let μ be a shift-invariant probability measure on Y. We put $\mathfrak{a} = Y^{\perp} \subset \mathfrak{R}_2^{(2)}$, $\mathfrak{M} = \mathfrak{R}_2^{(p)}/\mathfrak{a}$, observe that $\mathfrak{M}$ is a module over the ring $\mathfrak{R}_2^{(p)}$, and write $\hat{\mu}\colon \hat{Y} = \mathfrak{M} \longmapsto \mathbb{C}$ for the Fourier transform of μ, defined by $\hat{\mu}(a) = \int \langle a, x\rangle \, d\mu(x)$ for every $a \in \hat{Y}$. Then

$$\hat{\mu}(u^{\mathbf{n}} \cdot a) = \hat{\mu}(a)$$

for every $a \in \hat{Y}$ and $\mathbf{n} \in \mathbb{Z}^2$, and

$$\lim_{k\to\infty} \hat{\mu}\left(\sum_{\mathbf{n}\in E} u^{k\mathbf{n}} \cdot a(\mathbf{n})\right) = \prod_{\mathbf{n}\in E} \hat{\mu}(a(\mathbf{n}))$$

for every μ-mixing set $E \subset \mathbb{Z}^2$ and every map $\mathbf{n} \mapsto a(\mathbf{n})$ from E into $\hat{Y}$.

4. Lemma: *Let $E \subset \mathbb{Z}^2$ be a finite, non-empty set which is λ_Y-non-mixing. If μ is non-atomic, then E is μ-non-mixing.*

Proof: If μ is non-atomic, and if $0 \neq a \in \mathfrak{M} = \mathfrak{R}_2^{(p)}/\mathfrak{a}$, then we claim that the character $\chi_a = \langle a, \cdot\rangle$ of Y defined by a is not μ-a.e. equal to a constant. Otherwise $|\hat{\mu}(a)| = 1$, and the shift-invariance of μ implies that the set $\mathfrak{N} = \{a \in \mathfrak{M}\colon |\hat{\mu}(a)| = 1\} \subset \mathfrak{M}$ is a non-zero submodule, and that $Z = \mathfrak{N}^{\perp} \subset Y$ is a proper, closed, shift-invariant subgroup. By assumption, Z is finite, so that $\hat{Z} = \mathfrak{M}/\mathfrak{N}$ is finite. We set $\bar{\mu}(B) = \mu(-B)$ for every $B \in \mathfrak{B}_Y$ and conclude that the convolution $|\mu|^2 = \mu * \bar{\mu}$, whose Fourier transform is given by $\widehat{|\mu|^2} = |\hat{\mu}|^2$, is a probability measure with finite support. Hence μ has finite support, which is absurd.

Corollary 2.7 in [Kit-Sch2] implies that there exist elements $a_{\mathbf{n}} \in \mathfrak{M}, \mathbf{n} \in E$, not all equal to 0, such that $\sum_{\mathbf{n}\in E} u^{k\mathbf{n}} a_{\mathbf{n}} = 0$ for all k in an infinite subset $K \subset \mathbb{N}$. The first part of this proof shows that at last one of the characters $\chi_{a_{\mathbf{n}}}$ is non-constant μ-a.e. As $1 = \chi_{\sum_{\mathbf{n}\in E} u^{k\mathbf{n}} a_{\mathbf{n}}} = \prod_{\mathbf{n}\in E} \chi_{a_{\mathbf{n}}} \cdot \sigma_{k\mathbf{n}}^{Y}$ for every $k \in K$, the set E must be μ-non-mixing. ∎

Proof of Theorem 2: Let $E \subset \mathbb{Z}^2$ be a minimal λ_Y-non-mixing set which is also minimal μ-non-mixing, and choose as in the the proof of Lemma 4 a map $a\colon E \longmapsto \mathfrak{M} \smallsetminus \{0\}$ and an infinite subset $K \subset \mathbb{N}$ such that $\sum_{\mathbf{n}\in E} u^{k\mathbf{n}} a(\mathbf{n}) = 0$

for every $k \in K$. Then $u^{k\mathbf{n}} \cdot a(\mathbf{n}) = -\sum_{\mathbf{m} \in E \smallsetminus \{\mathbf{n}\}} u^{k\mathbf{m}} a(\mathbf{m})$ for every $k \in K$ and $\mathbf{n} \in E$. For every $\mathbf{n} \in E$, the set $E \smallsetminus \{\mathbf{n}\}$ is μ-mixing, and

$$\begin{aligned}\hat{\mu}(a(\mathbf{n})) &= \lim_{\substack{k\to\infty\\ k\in K}} \hat{\mu}(u^{k\mathbf{n}} \cdot a(\mathbf{n})) = \lim_{\substack{k\to\infty\\ k\in K}} \hat{\mu}\left(-\sum_{\mathbf{m}\in E\smallsetminus\{\mathbf{n}\}} u^{k\mathbf{m}} \cdot a(\mathbf{m})\right) \\ &= \prod_{\mathbf{m}\in E\smallsetminus\{\mathbf{n}\}} \overline{\hat{\mu}(a(\mathbf{m}))}.\end{aligned}$$

By varying $\mathbf{n}$ we obtain either that $|\hat{\mu}(a(\mathbf{n}))| = 1$ for every $\mathbf{n} \in E$, or that $\hat{\mu}(a(\mathbf{n})) = 0$ for every $\mathbf{n} \in E$.

As in the proof of Lemma 3 we set $\mathfrak{N} = \{a \in \mathfrak{M}: |\hat{\mu}(a)| = 1\}$ and observe that $\mathfrak{M}/\mathfrak{N}$ is finite and μ atomic whenever $\mathfrak{N} \neq \{0\}$. In particular, $\hat{\mu}(a(\mathbf{n})) = 0$ for every $\mathbf{n} \in E$, and by replacing the map $a: \mathbf{n} \mapsto a(\mathbf{n})$ from E to $\mathfrak{M}$ with $a': \mathbf{n} \mapsto a(\mathbf{n})' = h \cdot a(\mathbf{n})$ for an arbitrary, but temporarily fixed element $h \in \mathfrak{R}_2^{(p)}$ we see that $\hat{\mu}(h \cdot a(\mathbf{n})) = 0$ whenever $\mathbf{n} \in E$, $h \in \mathfrak{R}_2^{(p)}$, and $h \cdot a(\mathbf{n}) \neq 0$. Fix $\mathbf{n} \in E$ for the moment, and put $\mathfrak{N}' = \{h \cdot a(\mathbf{n}): h \in \mathfrak{R}_2^{(p)}\}$ and $Z' = \mathfrak{N}'^{\perp} \subset Y$. Since $\hat{\mu}(a) = 0$ for every non-zero $a \in \mathfrak{N}$, the measure μ' induced by μ on Y/Z' is equal to the Haar measure on Y/Z'. Since σ^Y is topologically conjugate to a skew-product action of $\mathbb{Z}^2$ on $Y/Z' \times Z'$, the ergodicity of μ under σ^Y implies that $\mu = \lambda_Y$. ∎

If we know a little more about the generator of the annihilator $Y^{\perp} = \mathfrak{a}$ of the subgroup Y, then we can weaken the assumptions of Theorem 2.

5. THEOREM: *Let $0 \neq f \in \mathfrak{R}_2^{(p)}$ be an irreducible Laurent polynomial, $\mathfrak{a} = f\mathfrak{R}_2^{(p)}$, and assume that the shift-action σ^Y of $\mathbb{Z}^2$ on $Y = \mathfrak{a}^{\perp} \subset X^{(p)}$ is mixing, and that the support $\mathcal{S}(f)$ of f is a minimal λ_Y-non-mixing set. If μ is a non-atomic, σ^Y-invariant probability measure on Y such that $\mathcal{S}(f) \smallsetminus \{\mathbf{n}\}$ is μ-mixing for some $\mathbf{n} \in \mathcal{S}(f)$, then $\mu = \lambda_Y$.*

Proof: Since $\sum_{\mathbf{n}\in\mathcal{S}(f)} c_f(\mathbf{n}) u^{p^k\mathbf{n}} \in \mathfrak{a}$ for every $k \geq 1$ we obtain that

$$\begin{aligned}\hat{\mu}(a) = \hat{\mu}(u^{p^k\mathbf{n}} \cdot a) &= \lim_{k\to\infty} \hat{\mu}\left(-\sum_{\mathbf{m}\in\mathcal{S}(f)\smallsetminus\mathbf{n}} \frac{c_f(\mathbf{m})}{c_f(\mathbf{n})} u^{p^k\mathbf{m}} \cdot a\right) \\ &= \prod_{\mathbf{m}\in\mathcal{S}(f)\smallsetminus\{\mathbf{n}\}} \overline{\hat{\mu}\left(\frac{c_f(\mathbf{m})}{c_f(\mathbf{n})} a\right)}\end{aligned}$$

for every $a \in \mathfrak{M} = \mathfrak{R}_2^{(p)}/\mathfrak{a}$. As $\mathbb{F}_p$ is finite and μ is non-atomic, we conclude as in the proof of Theorem 2 that $\hat{\mu}(a) = 0$ whenever $0 \neq a \in \mathfrak{M}$, and that $\mu = \lambda_Y$. ∎

6. *Examples:* In the following examples we consider irreducible Laurent polynomials $f \in \mathfrak{R}_2^{(2)}$ and set $\mathfrak{a} = f\mathfrak{R}_2^{(2)}$ and $Y = \mathfrak{a}^\perp \subset X^{(2)}$.

(1) (cf. [Kit-Sch2]) Let $f = 1 + u_1 + u_2 + u_1^2 + u_1u_2 + u_2^2$. Then $E = \{(0,0),(1,0),(0,1)\}$ is a minimal λ_Y-non-mixing set, and Theorem 2 implies that every σ^Y-invariant probability measure μ on Y for which $\sigma_{\mathbf{n}}^Y$ is μ-mixing for every $\mathbf{n} \in \{(1,0),(0,1),(1,-1)\}$, is equal to λ_Y.

(2) (cf. [Led], [Kit-Sch2]) Let $f = 1+u_1+u_2$. Then $\mathcal{S}(f) = \{(0,0),(1,0),(0,1)\}$ is minimal λ_Y-non-mixing, and by letting $\mathbf{n}$ vary in E we see from Theorem 5 that every probability measure μ on Y for which any of the transformations $\sigma_{(1,0)}^Y$, $\sigma_{(0,1)}^Y$, or $\sigma_{(-1,1)}^Y$ is mixing, must be equal to λ_Y. This example is of interest because of certain formal similarities between σ^Y and the $\mathbb{N}^2$-action on $\mathbb{T}$ generated by multiplication by 2 and 3 (cf. [Fur], [Rud]).

(3) Let $f = 1 + u_1 + u_1^{-1} + u_2$. Then $\mathcal{S}(f) = \{(-1,0),(0,0),(1,0),(0,1)\}$ is minimal λ_Y-non-mixing, and by setting $\mathbf{n} = (0,1)$ we see from Theorem 5 that every non-atomic, σ^Y-invariant probability measure μ on Y for which $\sigma_{(1,0)}^Y$ is three-mixing, is equal to λ_Y.

7. *Remark:* If $Y \subset X^{(p)}$ is a subgroup satisfying (a)–(c) then there exist shift-invariant, ergodic probability measures μ on Y which are different from λ_Y. A method for constructing such measures on the group Y in Example 6 (2) is described in [Kit-Sch1]. In general we put $\mathfrak{a} = Y^\perp$ and consider, for every $k \geq 1$, the subring $\mathcal{R}^{(k)} = \mathbb{F}_p[u_1^{\pm p^k}, u_2^{\pm p^k}] \subset \mathfrak{R}_2^{(p)}$. We write $\mathfrak{M}^{(k)}$ instead of $\mathfrak{M} = \mathfrak{R}_2^{(d)}/\mathfrak{a}$ in order to emphasize that $\mathfrak{M} = \mathfrak{M}^{(k)}$ is to be viewed as an $\mathcal{R}^{(k)}$-module. Then $\mathfrak{M}^{(k)}$ has non-trivial $\mathcal{R}^{(k)}$-submodules of infinite index. If $\mathfrak{N}^{(k)} \subset \mathfrak{M}^{(k)}$ is such a submodule, then $Y^{\mathfrak{N}^{(k)}} = (\mathfrak{N}^{(k)})^\perp \subset Y$ is an infinite, closed subgroup of Y which is invariant under the shifts $\{\sigma_{(mp^k,np^k)}\colon (m,n) \in \mathbb{Z}^2\}$. Although the Haar measure $\lambda^{\mathfrak{N}^{(k)}} = \lambda_{Y^{\mathfrak{N}^{(k)}}}$ is not shift-invariant, its orbit average $\mu^{\mathfrak{N}^{(k)}} = \frac{1}{p^{2k}} \sum_{m=0}^{p^k-1} \sum_{n=0}^{p^k-1} \lambda^{\mathfrak{N}^{(k)}} \cdot \sigma_{(m,n)}$ is a non-atomic, shift-invariant, ergodic probability measure on Y which is obviously non-mixing under every $\sigma_{\mathbf{n}}^Y$. By choosing sequences $\mathfrak{N}^{(1)} \subset \mathfrak{N}^{(2)} \subset \cdots$ of submodules $\mathfrak{N}^{(k)} \subset \mathfrak{M}^{(k)}$ we obtain sequences $(\mu^{\mathfrak{N}^{(k)}}, k \geq 1)$ of such measures, and every limit point μ of such a sequence is again shift-invariant. Note that these shift-invariant measures μ may

be chosen to have the property that the entropy $h_\mu(\sigma_{\mathbf{n}})$ is either positive or equal to 0 for some given non-zero $\mathbf{n} \in \mathbb{Z}^2$.

8. Examples: (1) Let $Y \subset X^{(p)}$ be a subgroup satisfying (a)–(c). For fixed $k \geq 0$, $l \geq 1$, let $f_{k,l} = 1 + u_1^{p^k} + u_1^{2p^k} + \cdots + u_1^{p^k(p^l-1)}$, and put $\mathfrak{N} = f_{k,l}\mathcal{R}^{(k+l)}/\mathfrak{a} \subset \mathfrak{M}^{(k+l)}$. Then $Y^{\mathfrak{N}} = \{y = (y_{\mathbf{n}}) \in Y\colon y_{(mp^{k+l},np^{k+l})} + y_{(mp^{k+l}+p^k,np^{k+l})} + \cdots + y_{(mp^{k+l}+p^k(p^l-1),np^{k+l})} = 0 \pmod p$ for every $(m,n) \in \mathbb{Z}^2\}$, and the orbit average $\mu^{\mathfrak{N}}$ of $\lambda_{Y^{\mathfrak{N}}}$ is non-atomic, shift-invariant, ergodic, and not equal to λ_Y.

(2) Let $f = 1 + u_1 + u_2^2 \in \mathfrak{R}_2^{(2)}$, $\mathfrak{a} = f\mathfrak{R}_2^{(2)}$, and $Y = \mathfrak{a}^\perp \subset X^{(2)}$. Then $\mathfrak{M} = \mathfrak{R}_2^{(2)}/\mathfrak{a}$ is a module over the ring $\mathcal{R} = \mathbb{F}_2[u_1^{\pm1}, u_2^{\pm2}]$, and $\mathfrak{L} = (1+u_2)\mathcal{R}/\mathfrak{a} \subset \mathfrak{M}$ is an $\mathcal{R}$-submodule with infinite index. Put $Y^{\mathfrak{L}} = \mathfrak{L}^\perp = \{y = (y_{\mathbf{n}}, \mathbf{n} \in \mathbb{Z}^2)\colon y_{(2m,n)} = y_{(2m+1,n)}$ for all $(m,n) \in \mathbb{Z}^2\}$, and note that $\mu = \frac{1}{2}(\lambda_{Y^{\mathfrak{L}}} + \lambda_{Y^{\mathfrak{L}}} \cdot \sigma^Y_{(0,1)})$ is non-atomic, shift-invariant and ergodic, but non-ergodic under $\sigma_{(1,0)}$.

9. Problem: If $Y \subset X^{(p)}$ is a subgroup satisfying (a)–(c), can there exist a non-atomic, shift-invariant probability measure $\mu \neq \lambda_Y$ on Y which is mixing under some $\sigma^Y_{\mathbf{n}}$, $\mathbf{n} \in \mathbb{Z}^2$?

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